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NUMERICAL SOLUTIONS FOR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS VIA ACCELERATED GENETIC ALGORITHM

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ABSTRACT

This paper introduced a new accelerated Genetic Algorithms (GAs) method to find a numerical solutions of stochastic Partial differential equations driven by space-time white nose wiener process . The numerical scheme is based on a representation of the solution of the equation involving a stochastic part arising from the noise and a deterministic partial differential equation . By using Doss-Sussmann transformation that enables us to work with a partial differential equation instead of the stochastic partial differential equation. Then compare these solutions obtained by our method with saul'yev method and deterministic solution.

KEYWORDS: SPDS, Accelerated Genetic Algorithm method, Numerical solution of stochastic partial differential equations.

INTRODUCTION

In this paper we want to take a quicker look at the numerical solutions for stochastic partial differential equations (SPDEs). Working on the numerical solutions for SPDEs we face many difficulties. On the one hand we have to consider problems known from numerically solving deterministic partial differential equations. On the other hand we are faced with problems triggered by numerically solving stochastic ordinary differential equations (SODEs). And additionally new issues arise resulting from the infinite dimensional nature of the underlying noise processes ,[1].

Stochastic partial differential equations (SPDEs) are used as a model in many applications. This area of mathematics is especially motivated by the need to describe random phenomena studied in natural sciences like physics, chemistry, biology, and in control theory, [2]. So, we can define SPDEs, by combine deterministic partial differential equations with some kind of noise.

Consider the SPDE with space-time white noise ,[4].

$$du(t,x) = u_{xx}(t,x)dt + g(u(t,x))dW(t,x) \qquad (1)$$

with $0 < x < 1$, and

$$u(0,x) = u_0 \ , u(t,0) = u(t,1) = 0 \ , and \ \ u_t(t,0) = u_t(0,1) = 0$$

Then , two different ways of writing this equation (1) are :

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(u)\frac{\partial^2 W}{\partial t \partial x} \qquad (2)$$

or

$$du = u_{xx}dt + \sum_{k=1}^{\infty} g(u)h^k(x)dW_k(t) \qquad (3)$$

Such that , three kinds of space-time white noise as in [4] are :

- ***Brownian Sheet*** – $W(t,x) = \mu([0,T] \times [0,x])$
- ***Cylindrical Brownian motion*** – family of Gaussian random variables

$B_t = B_t(h)$, $h \in H$ a Hilbert space, s.t.
$E[B_t(h)] = 0, \quad E[B_t(h)B_s(g)] = \langle h, g\rangle_H \, (t \wedge s)$

- ***Space-time white noise*** $dW(t,x) = \frac{\partial^2 W}{\partial t \partial x} = \sum_{k=1}^{\infty} h^k(x) dW_k(t)$, where $\{h^k\}$ is assumed a Basis of the Hilbert space we're in , if $\{h^k, k > 1\}$ is a complete orthonormal system, then $\{B_t(h^k), \ k > 1\}$ independent standard Brownian motion.

The connection between the three kinds: If $H = L^2(\mathbb{R})$ *or* $H = L^2(0,1)$, then

$$B_t(h) = \int \frac{\partial W}{\partial x} h(x) dx$$

and

$$B_t(x) = B_t(X_{[0,x]}) = \sum_{k=1}^{\infty} \int_0^x (h^k(y) dy W_k(t)) = W(t,x)$$

where we assume that

$$h^k(x) = \sqrt{2}\sin(k\pi x) \tag{4}$$

Then we get equations of the form :

$$dU(t,x) = [\mathcal{A}U(t,x) + c(x)U(t,x)]dt + \sum_{l=1}^{\infty} h^l(t)U(t,x)dB_t^l \tag{5}$$

Or in integral form

$$U(t,x) = U_0(x) + \int_0^t \mathcal{A}U(s,x)ds + \int_0^t c(x)U(s,x)ds + \sum_{l=1}^{n} \int_0^t h^l(s)U(s,x)dB_s^l \tag{6}$$

For $0 \le t \le T$. The process $B = (B_t^l, \dots, B_t^n)_{0 \le t \le T}$ is an n-dimensional Brownian motion . The operator $\mathcal{A}$ is defined as :

$$\mathcal{A}u = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} \quad , \qquad with \ a_{ij}(x) = \sum_{k=1}^{m} \sigma_{ik}\sigma_{jk} \tag{7}$$

Where the diffusion matrix $\sigma: \mathbb{R}^{\boldsymbol{d}} \to \mathbb{R}^{\boldsymbol{d \times m}}$ and the drift coefficient $b: \mathbb{R}^{\boldsymbol{d}} \to \mathbb{R}^{\boldsymbol{d}}$. the initial condition U_0 , the functions c, σ *and* b are suppose to be smooth functions of the space variable , $(h^l(t))_{1 \le l \le n}$ are bounded and holder continuous of order 1/2. Thus the equation (5) has a unique regular strong solution.

In this paper, we focus on the stochastic heat equation. Thus, we simplify the above equation to :

$$dU(t,x) = \mathcal{A}U(t,x)dt + \mathcal{S}\big(U(t,x)\big)dW(t,x) \tag{8}$$

where $\mathcal{S}$ is a multiplication operator of the form

$$(\mathcal{S}(v)u)(x) = b\big(x, v(x)\big).u(x)$$

Taking a closer look at the noise in this equation we see that we can split it into two types, *additive* and *multiplicative noise*. We speak of *additive noise* if the operator $\mathcal{S}$ is a constant operator and of *multiplicative noise* if $\mathcal{S}$ is not constant.

The objective of our work is to develop a numerical scheme for the random field U. The problem of numerical solutions of (5) has been studied by many authors with different approaches. The ideas that lead us to propose a new scheme are twofold.

- On the one hand we wish to propose a numerical scheme that separates the noise $\mathcal{S}$ from the second order operator $\mathcal{A}$. This idea has been used in [2] ,[5] in a filtering context in which the authors' scheme first performs off-line a wide number of solutions of partial differential equations by the finite element method. The stochastic part of the simulation is done after this first step.
- On the other hand we want to use the accelerated genetic algorithm method to find numerical solution for the partial differential equations that may appear in our scheme.

In order to implement the above ideas, we need to introduce the d-dimensional Markov process $X = (X_t)_{0 \le t \le T}$ whose infinitesimal generator is given by the second order operator in equation (7) .

The Markov process X is governed by infinitesimal generator $\mathcal{A}$ of the stochastic differential equation is :

$$X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)\, dW_s \quad ,0 \le t \le T \qquad (10)$$

where the initial condition x_0 in $\mathbb{R}^d$.

TYPES OF SOLUTIONS OF SPDES

Stochastic partial differential equations of the form

$$dX_t(x) = [\mathcal{A}X_t + F(X_t)]dt + \mathcal{S}(X_t)dW_t(x) \quad ,X(0) = \xi \qquad (11)$$

have different notions of solutions. As in [1] we find :

Definition 1: $D(\mathcal{A})$-valued predictable process $X(t)$, $t \in [0\,,T]$ is called *an analytical strong solution* of the problem (6) if

$$X(t) = \int_0^t [\mathcal{A}X_s + F(X_s)]ds + \int_0^t \mathcal{S}(X_s)dW_s \quad ,P - a.s. \qquad (12)$$

In particular ,the integral in the right-hand side have to be well-define ,[1],[4].

Definition 2: H-valued predictable process $X(t)$, $t \in [0\,,T]$ is called *an analytical weak solution* of the problem (6) if

$$< X(t),\zeta >= \int_0^t [< X(s),\mathcal{A}^*\zeta > + < F(X_s),\zeta >]ds + \int_0^t < \zeta,\mathcal{S}(X_s)dW_s > \,,P - a.s. (13)$$

For each $\zeta \epsilon D(\mathcal{A}^*)$, in particular ,the integral in the right-hand side have to be well-define.

Definition 3: H-valued predictable process $X(t)$, $t \in [0\,,T]$ is called an *mild solution* of the problem (6) if

$$X(t) = \int_0^t [e^{\mathcal{A}(t-s)}F(X_s)]ds + \int_0^t e^{\mathcal{A}(t-s)}\mathcal{S}(X_s)dW_s \quad ,P - a.s. \qquad (14)$$

In particular ,the integral in the right-hand side have to be well-define,[1],[4].

STOCHASTIC INTEGRAL WITH RESPECT TO CYLINDRICAL WIENER PROCESS

We denote by $L_0^2 = L^2(U_0, Y)$ the space of Hilbert-Schmidt operators acting from U_0 into Y, and by $L = L(U,Y)$, we denote the space of linear bounded operators from U into Y ,[1],[4].

Let us consider the norm of the operator $\psi \in L_2^0$:

$$\|\psi\|_{L_0^2}^2 = \sum_{h,k=1}^{\infty} < \psi g_h, f_k >_Y^2 = \sum_{h,k=1}^{\infty} \lambda_h < \psi e_h, f_k >_Y^2 = \left\|\psi Q^{\frac{1}{2}}\right\|_{HS}^2 = tr(\psi Q \psi^*) \ (15)$$

Where $g_i = \sqrt{\lambda_i} e_i$ and $\{\lambda_i\},\{e_i\}$ are eigenvalues and eigenfunctions of the operator Q , $\{g_i\}\{e_i\}$ and $\{f_i\}$ are orthonormal bases of spaces U_0, U and Y , respectively. The space L_2^0 is a separable Hilbert space with the norm

$$\|\psi\|_{L_0^2}^2 = tr(\psi Q \psi^*) \qquad (16)$$

In particular

1.When $Q = I$ then $U_0 = U$ and the space L_0^2 becomes $L^2(U,Y)$.

2. When Q is a nuclear operator, that is $trQ < +\infty$, then $L(U,Y) \subset L^2(U_0,Y)$. For, assume that $K \in L(U,Y)$, that is K is linear bounded operator from the space U into Y.

Let us consider the operator $\psi = K|_{U_0}$, that is the restriction of operator K to the space U_0, where $U_0 = Q^{\frac{1}{2}}(U)$. Because Q is nuclear operator, then $Q^{\frac{1}{2}}$ is Hilbert- Schmidt operator.

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Proposition.1 the formula

$$W_c(t) = \sum_{j=1}^{\infty} g_j \beta_j(t) \quad , t \geq 0 \qquad (17)$$

defines Wiener process in U_1 with covariance operator Q_1 such that $trQ_1 < +\infty$.

Proposition.2 For any $a \in U$ the process

$$< a, W_c(t) >_U = \sum_{j=1}^{\infty} < a, g_j >_U \beta_j(t) \qquad (18)$$

is real-valued Wiener process and

$$E < a, W_c(t) >_U < b, W_c(t) >_U = (t \wedge s) < Qa, b >_U \ , a, b \in U$$

Additionally, $ImQ_1^{\frac{1}{2}} = U_0$ and $\|u\|_{U_0} = \left\| Q_1^{-\frac{1}{2}} u \right\|_{U_1}$.

In the case when Q is nuclear operator, $Q^{\frac{1}{2}}$ is Hilbert-Schmidt operator. Taking $U_1 = U$, the process $W_c(t)$, $t \geq 0$, defined by (17) is the classical Wiener process introduced.

Definition.4 The process $W_c(t)$, $t \geq 0$, defined in (17), is called *cylindrical Wiener process* in U when $trQ_1 < +\infty$. As shown in Fig.1 below .

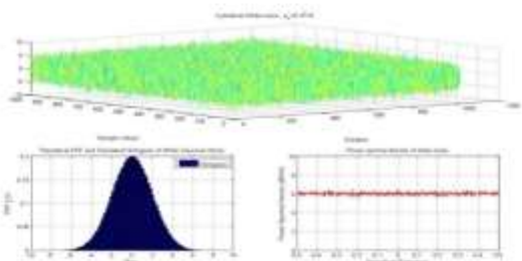

Fig.1 Cylindrical White noise with its distribution and spectral density

The stochastic integral with respect to cylindrical Wiener process is defined as follows. As we have already written above, the process $W_c(t)$, $t \geq 0$ defined by (10) is a Wiener process in the space U_1 with the covariance operator Q_1 such that $trQ_1 < +\infty$.
Then the stochastic integral ,

$$\int_0^t g(s) dW_c(s) \in Y \qquad (19)$$

where $g(s) \in L(U_1, Y)$, with respect to the Wiener process $W_c(t)$ is well defined on U_1.
We denote by $N(Y)$ the space of all stochastic processes

$$\emptyset: [0, T] \times \Omega \to L^2[U_0, Y]$$

Such that

$$E\left(\int_0^T \|\emptyset(t)\|^2_{L^2[U_0,Y]} dt\right) < +\infty \qquad (20)$$

and for all $u \in U_0$, $\emptyset(t)u$ is a Y-valued stochastic process measurable with respect to the filtration $\mathcal{F}_t$.
The stochastic integral

$$\int_0^t \emptyset(s) dW_c(s) \in Y$$

with respect to cylindrical Wiener process, given by (10) for any process $\emptyset \in N(Y)$ can be defined as the limit

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$$\int_0^t \emptyset(s)dW_c(s) = \lim_{m\to\infty}\sum_{j=1}^{m}\int_0^t \emptyset(s)g_j d\beta_j(s) \quad in\ Y \qquad (21)$$

In $L_2(\Omega)$ sense .

MATHEMATICAL SETTING AND ASSUMPTIONS ,[1]

Let $T > 0$ and let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space with a normal filtration $\mathcal{F}_t, t \in [0,T]$. in addition let $(H, <.,.>)$ be a separable Hilbert space with norm denoted by $|.|$ .We will interpret the SPDE (1) in such a space H . The objects $\mathcal{A}$, x_0 , F, W_t , here are specified through the following assumptions.

Assumption 1: *linear operator* $\mathcal{A}$**.** There exist sequence of real eigenvalues $0 \le \lambda_1 \le \lambda_2 \le \cdots$ and eigenfunctions $\{e_n\}_{n\ge1}$ of $\mathcal{A}$ such that the linear operator $A: D(\mathcal{A}) \subset H \to H$ is given by :

$$\mathcal{A}v = \sum_{n=1}^{\infty} -\lambda_n < e_n, v)e_n , \qquad (22)$$

For all

$$v \in D(A) with\ D(\mathcal{A}) = \left\{ v \in H | \sum_{n=1}^{\infty} |\lambda_n|^2 | < e_n, v > |^2 < \infty \right\}$$

Let $D((-\mathcal{A})^r)\ with\ r \in \mathbb{R}$ denote the interpolation space of the operator $(-\mathcal{A})$,[8].

Assumption 2: *Cylindrical Brownian motion* W_t **.** there exist a sequence of $q_n \ge 0$, $n \ge 1$, of positive real numbers $\gamma \in (0,1)$ such that

$$\sum_{n=1}^{\infty} (\lambda_n)^{2\gamma-1} q_n < \infty \qquad (23)$$

And independent real valued $\mathcal{F}_t - Brownian\ motion\ \beta_t^n, t \ge 0, n \ge 1$, i.e. each β_t^n is $\mathcal{F}_t$-adapted and the increments $\beta_{t+h}^n - \beta_t^n$, $h > 0$, are independent of $\mathcal{F}_t$. Then the cylindrical Brownian motion W_t is given by :

$$W_t(x) = \sum_{n=1}^{\infty} \sqrt{q_n} e_n(x)\beta_t^n \qquad (24)$$

Remark 1. The above series may not converge in H, but in some space U_1 into which H can be embedded, ([7] and [8]). In our example with the Laplace operator in one dimension, we will have $\lambda_n = -\pi^2 n^2$ and $q_n \equiv 1$, $for\ n \ge 1$. This is the important case of space–time white noise.

Assumption 3: *nonlinearity* f. The nonlinearity $f: H \to H$ is two times continuously differentiable, it and its derivatives satisfy

$$|f'(x) - f'(y)| \le L|x - y|, \qquad |(-\mathcal{A})^{(-r)} f'(x)(-\mathcal{A})^r v| \le L|v|$$

For all $x, y \in H$, $v \in D(-\mathcal{A})^r\ and\ r = 0, \frac{1}{2}, 1$ and they satisfy

$$|\mathcal{A}^{-1} f''(x)(v,w)| \le L \left|(-\mathcal{A})^{-\left(\frac{1}{2}\right)} v\right| \left|(-\mathcal{A})^{-\left(\frac{1}{2}\right)} w\right|, for\ all\ v, w, x \in H, L > 0$$

Remark 2. The function f is usually given as a real-valued function of a real variable, but in the SPDE (1) it is considered as a function defined on H and taking values in some function space such as a subspace of H.
Assumption 4: *initial value* X_0. The initial value $X_0$is a $D((-\mathcal{A})^r)$ valued random variable, which satisfies

$$E|(-\mathcal{A})^{-\left(\frac{1}{2}\right)} X_0|^4 < \infty \qquad (25)$$

where $\gamma > 0$ is given in assumption 2.2.
With the above assumptions we get by [JK11] that

$$dX_t = [k\Delta X_t + f(x, x_t)]dt + b(x, X_t)dW_t(x) \qquad (26)$$

With

$$X_0(x) = 0 \quad and \quad X_t(0) = X_t(1) = 0 \qquad (27)$$

$for\ x \in (0,1), t \in [0,T)$ has unique mild solution

$$X: [0,T] \times \Omega \to H_{\beta+\frac{1}{2}} \qquad (28).$$

A RELATED PARTIAL DIFFERENTIAL EQUATION

We present in this section a transformation that enables us to work with a partial differential equation instead of the stochastic partial differential equation (5). This method is classical and it is known as the Doss-Sussmann transform when one applies it to stochastic differential equation ([7] and [8]). It is a useful trick that permits to rewrite a large class of one dimensional stochastic dynamic as a one dimensional random ordinary dynamic (by stochastic dynamic we mean stochastic differential equation or stochastic partial differential equation). It has been successfully used in [8] in which the authors have estimated the probability of finite-time blowup of positive solutions of stochastic partial differential equations with Dirichlet boundary condition.

Doss-Susmann transform

The particular form of (5) will allow us to use a Doss-Susmann transform. We may write that ,[5].

$$U(t,x) = \exp\left(\sum_{l=1}^{n} \int_0^t h^l(s) dB_s^l\right) \times v(t,x) \qquad (29)$$

with v that solves the partial differential equation

$$dv(t,x) = \mathcal{A}v(t,x)dt + \left(c(x) - \frac{1}{2}\sum_{l=1}^{n} \left(h^l(t)\right)^2\right) v(t,x)dt \qquad (30)$$

As regard to the expression of the function v, it is clear that v can be simulated off-line. Indeed the coefficients in the above partial differential equation are $c, (h^l)_{1<l<n}$ and the coefficients of the Markov process X. They are all supposed to be known. Consequently, we can perform a wide number of computations related to the partial differential equation satisfied by v. Then we shall come back to the simulation of U itself and we use as much as we want the previous computations. Thus we have split our scheme into a deterministic part (the approximation of v) and a stochastic part (the immediate computation of U when one simulates the Brownian motion B). The approximation of v and the Markov process X will be achieved by a accelerated genetic algorithm . We have the following proposition.

Proposition 3. Let u be the solution of (5). Then the function v defined almost-surely by

$$v(t,x) = u(t,x)\, exp\left(-\sum_{l=1}^{n} \int_0^t h^l(s) dB_s^l\right) \qquad (31)$$

is the unique strong solution of the following parabolic partial differential equation

$$dv(t,x) = \mathcal{A}v(t,x)dt + \left(c(x) - \frac{1}{2}\sum_{l=1}^{n} \left(h^l(t)\right)^2\right) v(t,x)dt \qquad (32)$$

$0 \le t \le T$, The above equation is understood trajectory wise since it is valid for almost-all ω.

Proof. We denote $E = (E_t)_{0\le t\le T}$ the process defined by

$$E_t = exp\left(-\sum_{l=1}^{n} \int_0^t h^l(s) dB_s^l\right) \qquad (33)$$

It is a semi-martingale with the decomposition

$$E_t = 1 - \sum_{l=1}^{n} \int_0^t h^l(s) E_s dB_s^l + \frac{1}{2}\sum_{l=1}^{n} \int_0^t (h^l(s))^2 E_s ds \qquad (34)$$

In view of (1), for all $x \in R^d$, $(u(t,x))_{0\le t\le T}$ is a semi-martingale and we have

$$\langle E, u(.,x)\rangle = -\sum_{l=1}^{n} \int_0^t \left(h^l(s)\right)^2 E_s u(s,x) ds = -\sum_{l=1}^{n} \int_0^t \left(h^l(s)\right)^2 v(s,x) ds \qquad (35)$$

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Since E_t does not depend on the space variable *x*, it holds that

$$E_t\mathcal{A}u(t,x) = \mathcal{A}\big(E_t u(t,x)\big) = \mathcal{A}v(t,x)$$

and the integration by parts formula yields the result.

■

ACCELERATED GENETIC ALGORITHM

The principles of genetic algorithm are discussed in previous paper [9]. Where the components of the genetic algorithm, ,[10],[11],[12]are :

1. Initialization

The value of mutation rate and selection rate are stated ,[9]. The initialization of every chromosome is performed by randomly selecting an integer for every element of the corresponding vector.

2. Fitness evaluation

Expressing the Partial differential equation in the following form:

$$f\left(x, y, \frac{\partial u}{\partial x}(x,y), \frac{\partial u}{\partial y}(x,y), \frac{\partial^2 u}{\partial x^2}(x,y), \frac{\partial^2 u}{\partial y^2}(x,y)\right) = 0 \qquad (36)$$

$$x \in [x_0, x_1] \quad , \quad y \in [y_0, y_1]$$

The associated boundary conditions are expressed as:

$$u(x_0\,, y) = f_0(y) \quad , u(x_1\,, y) = f_1(y) \quad , \quad u(x\,, y_0) = g_0(y) \quad , \quad u(x\,, y_1) = g_1(y) \quad (37)$$

The steps for the fitness evaluation of the population are the following:

1. Choose N^2 equidistant points in the box $[x_0, x_1] \times [y_0, y_1]$, N_x equidistant points on the boundary at $x = x_0$ and at $x = x_1$, N_y equidistant points on the boundary at $y = y_0$ and at $y = y_1$

2. For every chromosome i :

(i) Construct the corresponding model $M_i(x\,, y)$, expressed in the grammar described earlier.

(ii) Calculate the quantity

$$E(M_i) = \sum_{j=0}^{N^2}(f(x_j, y_j, \frac{\partial}{\partial x}M_i(x_j, y_j), \frac{\partial}{\partial y}M_i(x_j, y_j), \frac{\partial^2}{\partial x^2}M_i(x_j, y_j), \frac{\partial^2}{\partial y^2}M_i(x_j, y_j))^2 \quad (38)$$

(iii) Calculate an associated penalty $P_i(M_i)$. The penalty function P depends on the boundary conditions and it has the form:

$$P_1(M_i) = \sum_{j=1}^{N_x}(M_i(x_0, y_j) - f_0(y_j))^2$$
$$P_2(M_i) = \sum_{j=1}^{N_x}(M_i(x_1, y_j) - f_1(y_j))^2 \qquad (39)$$
$$P_3(M_i) = \sum_{j=1}^{N_y}(M_i(x_j, y_0) - g_0(x_j))^2$$
$$P_4(M_i) = \sum_{j=1}^{N_y}(M_i(x_j, y_1) - g_1(x_j))^2$$

(iiii) Calculate the fitness value of the chromosome as:

$$v_i = E(M_i) + P_1(M_i) + P_2(M_i) + P_3(M_i) + P_4(M_i) \qquad (40)$$

3. Genetic operators

The genetic operators that are applied to the genetic population are the initialization, the crossover and the mutation. A random integer of each chromosome was selected to be in the range [0..255] . The parents are selected via tournament selection, i.e. :

- First, create a groups of $K \geq 2$ randomly selected individuals from the current population.
- The individuals with the best fitness in the group is selected, the others are discarded.

The final genetic operator used is the mutation, where for every element in a chromosome a random number in the range [0 , 1] is chosen,[9].

4. Termination control

Creating new generation required for application genetic operators to the population in order to find the best chromosome having better fitness or whenever the maximum number of generations was obtained.

5. Technical of the Accelerated Method

To make the method is faster to arrived the exact solution of the partial differential equations by the following :

1- Insert the boundary conditions of the partial differential equation as a part of chromosomes in the our population of the problem, the algorithm gives the exact solution or numerical approximation solution in a few generations.
2- Insert a part of exact solution (or particular solution) as a part of a chromosome in the population, find the algorithm that gives an exact solution in a few generations.
3- Insert the vector of exact solution (if exist) as a chromosome in the our population of the problem, the algorithm gives the exact solution in the first generation.

APPLICATION OF THE ACCELERATED GENETIC ALGORITHM

In this section we applied our algorithm on some SPDEs driven by cylindrical Brownian motion with additive and multiplicative cases.

1. Stochastic Partial differential equations with additive noise.

We first look at SPDEs with additive noise to get a reference about how well the earlier presented method work. We consider the stochastic heat equation with additive space–time white noise on the one-dimensional domain [0,1] over the time interval $[0,T]$ with $T=1$.

Consider the following SPDE

$$dX_t(x) = \left[\kappa\Delta X_t(x) + f\left(x, X_t(x)\right)\right]dt + b\left(x, X_t(x)\right)dW_t(x) \qquad (41)$$

with $X_0(x) = 0$ and $X_t(0) = X_t(1) = 0$ for $x \in (0,1), t \in [0,T)$.

and $f(x,y) = 0$, $b(x,y) = 1$, where the noise $W_t(x)$ here is the space-time white noise wiener process

$$W_t(x) = \sum_{n=1}^{\infty} e_n(x)\beta_t^n$$

with $q \equiv 1$ for all $n \geq 1$ in view of assumption 2.2. (The summation here is just formal, it does not converge in H.) Therefore, we have $\gamma = \left(\frac{1}{4}\right) - \varepsilon$,with an arbitrary small $\varepsilon > 0$ in our situation.

Then the SPDE

$$dX_t(x) = [\kappa\Delta X_t(x)\,]dt + dW_t(x) \qquad (42)$$

has unique mild solution $X: [0,T] \times \Omega \to H_{\beta+\frac{1}{2}}$. where described in [Kru12] can be written as

$$X_t = \int_0^t e^{\mathcal{A}(t-s)} dW_s = \sum_{n=1}^{\infty} e_n \int_0^t e^{-\lambda_n(t-s)} d\beta_s \quad (43)$$

where we use the eigenvalues

$$\lambda_n = n^2\pi^2 \qquad (44)$$

and eigenvectors

$$e_n(x) = \sqrt{2}\sin(n\pi x) \qquad (45)$$

for all $n \geq 1$ of the operator $\mathcal{A}$.

<u>Example 1</u> Let we try to find the numerical solution of the SPDE with additive noise.

$$dU_t(x) = \left[\frac{\partial^2 U(t,x)}{dx^2}\right]dt + dW_t(x) \qquad (46)$$

With

$$U_0(x) = 0 \quad and \quad U_t(0) = U_t(1) = 0 \qquad (47\,)$$

for $x \in (0,1), t \in [0,T]$ where $W_t(x)$ is space-time white noise wiener process.

By using Doss-Susmann transform (30). We find

$$dv(t,x)=\frac{\partial^2 v}{\partial x^2}(t,x)dt+\left(0-\frac{1}{2}\sum_{l=1}^{n}\left(h^l(t)\right)^2\right)v(t,x)dt \qquad (48)$$

with

$$\mathcal{A}U=\frac{1}{2}\sum_{i,j=1}^{d}a_{ij}\frac{\partial^2 U}{\partial x_i\partial x_j}+\sum_{i=1}^{d}b_i\frac{\partial U}{\partial x_i}=\sum_{i,j=1}^{1}\frac{\partial^2 U}{\partial x_i\partial x_j}+\sum_{i=1}^{1}0\frac{\partial U}{\partial x_i}=\frac{\partial^2 U}{\partial x^2} \qquad (49)$$

Then $b(X_t)=0\ and\ \sigma_{ij}=\sqrt{2}$,and the Markov process X was governed by the operator $\mathcal{A}$ of this stochastic partial differential equation is:

$$X_t=x_0+\int_0^t\sqrt{2}\,dB_s \quad ,0\le t\le T \qquad (50)$$

where the initial condition x_0 in $\mathbb{R}^{\boldsymbol{d}}$. if $h(t)=1$ and $n=1$, then (48) became :

$$\frac{dv(t,x)}{dt}=\frac{\partial^2 v}{\partial x^2}(t,x)-\frac{1}{2}v(t,x) \qquad (51)$$

With $v_0(x)=0\ and\ v_t(0)=v_t(1)=0\ for\ x\in(0,1),t\in[0,T)$

Now find the numerical solution of the partial differential equation (PDE)(51) by using an accelerated genetic algorithm. We found that

$$Gp(t,x)=\exp\left(-\frac{3}{2}t\right)sinx \qquad (52)$$

And the solution of the stochastic ordinary differential equation (50) (Markov process) generated by the infinitesimal generator A by accelerated genetic algorithm is :

$$X_t=x_0+\sqrt{2}B(t) \qquad (53)$$

Then , the solution of the original equation (46) is obtained by substituting (52),(53) in equation (29):

$$U(t,x)=\exp\left(\sum_{l=1}^{n}\int_0^1 h^l(s)dB_s^l\right)\times GP(t,X_t)=\ \exp\left(\sum_{l=1}^{n}\int_0^1 h^l(s)dB_s^l\right)\times\exp\left(-\frac{3}{2}t\right)\sin\left(x_0+\sqrt{2}B(t)\right) \qquad (54)$$

Fig.1 shown this solution

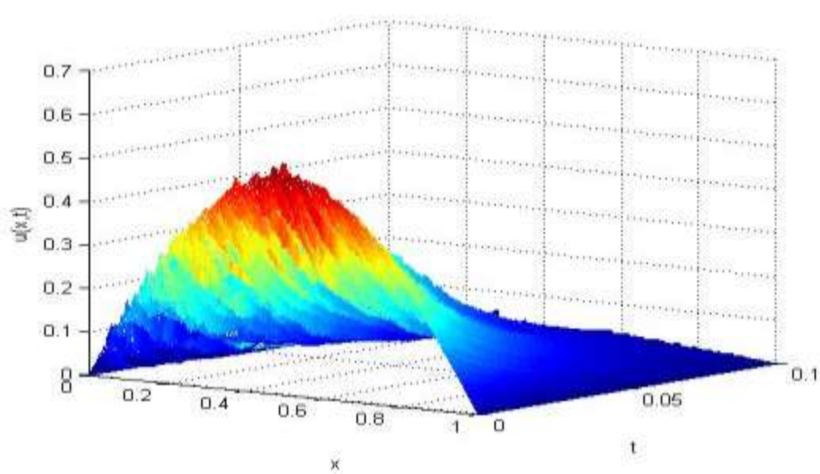

Fig.1 solution of SPDE (46)

and then compared this solution by our method with the solution obtained by Saul'yev method ,[13]. And with its corresponding deterministic solution. (In this problem and other test examples, by a deterministic solution we mean the numerical solution of the unperturbed problems). This comparison shown in Fig. 2 below :

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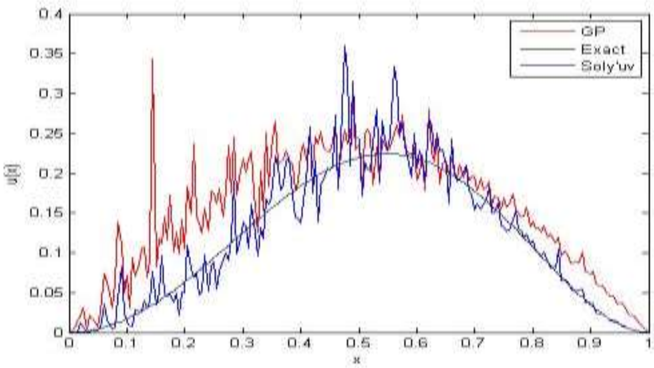

Fig.2 comparison of solutions of SPDE (46)

2. Stochastic Partial differential equations with multiplicative noise.

Look at SPDEs with multiplicative noise to get a reference about how well the earlier presented method work. We consider the stochastic heat equation with multiplicative space–time white noise on the one-dimensional domain [0,1] over the time interval $[0, T]$ with $T = 1$.

Example 2 Let us try to find the numerical solution of the SPDE with multiplicative noise.
Consider the SPDE

$$dU_t = \kappa\Delta U_t(x)dt + U_t(x)dW_t(x) \qquad (55)$$

With

$$U_0(x) = x^2(1 - \sin\left(\frac{\pi}{2}x\right)^2) \ and \ \ U_t(0) = U_t(1) = 0 \qquad (56)$$

$for\ x \in (0,1), t \in [0, T)$ where $W_t(x)$ is space-time white noise wiener process . where κ is a small parameter, we will have $\kappa = \frac{1}{100}$. By using Doss-Susmann transform (30) . We find

$$dv(t,x) = \frac{\partial^2 v}{\partial x^2}(t,x)dt + \left(0 - \frac{1}{2}\sum_{l=1}^{n}\left(h^l(t)\right)^2\right)v(t,x)dt \qquad (57)$$

With

$$\mathcal{A}U = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\frac{\partial^2 U}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i\frac{\partial U}{\partial x_i} = \sum_{i,j=1}^{1}\frac{1}{100}\frac{\partial^2 U}{\partial x_i \partial x_j} + \sum_{i=1}^{1} 0\frac{\partial U}{\partial x_i} = \frac{1}{100}\frac{\partial^2 U}{\partial x^2} \qquad (58)$$

Then $b(X_t) = 0\ and\ \sigma_{ij} = \sqrt{0.02}$,and the Markov process X was governed by the infinitesimal generator of this stochastic differential equation $\mathcal{A}$ is:

$$X_t = x_0 + \int_0^t \sqrt{0.02}\, dB_s \ \ , 0 \le t \le T \qquad (59)$$

where the initial condition x_0 in $\mathbb{R}^{\boldsymbol{d}}$. if $h(t) = 1$ and $n = 1$, then (57) became :

$$\frac{dv(t,x)}{dt} = 0.01\frac{\partial^2 v}{\partial x^2}(t,x) - \frac{1}{2}v(t,x) \qquad (60)$$

With

$$v_0(x) = x^2(1 - \sin\left(\frac{\pi}{2}x\right)^2) \ \ and\ v_t(0) = v_t(1) = 0 \qquad (61)$$

$for\ x \in (0,1), t \in [0, T)$.

Now find the numerical solution of the partial differential equation (PDE)(60) by using accelerated genetic algorithm. We found that at generation 26, the numerical solution is:

$$Gp26(t,x) = 2e^{-t}sin(3x^2) \qquad (62)$$

and the solution of the stochastic ordinary differential equation (59) (Markov process) generated by the infinitesimal generator $\mathcal{A}$ by accelerated genetic algorithm is :

$$X_t = x_0 + \sqrt{0.02}B(t) \qquad (63)$$

Then , the solution of the original equation (55) is obtained by substituting (62),(63) in equation (29) , we find :

$$U(t,x) = e^{W_t(x)} \times GP26(t, X_t) = e^{W_t(x)} \times 2e^{-t} \sin\left(3(x_0 + \sqrt{0.02}B(t))^2\right) \qquad (64)$$

Fig.3 show this solution

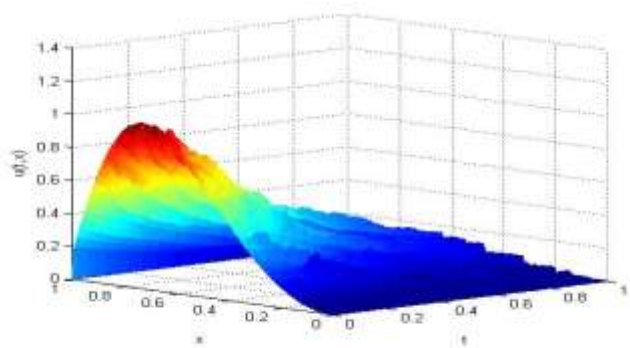

Fig.3 The solution of SPDE (55)

and then compared this solution by our method with the solution obtained by Saul'yev method and with its corresponding deterministic solution. This comparison shown in Fig. 4 below :

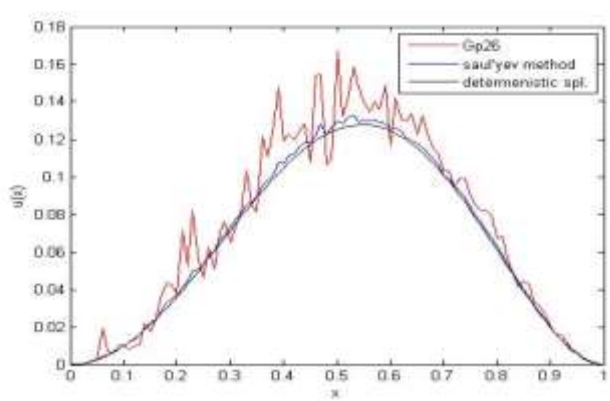

Fig.4 comparison of solutions of SPDE (55)

The comparisons of errors of these solutions was shown in table (4.1).

Table (4.1) Comparisons of the errors.

t	***x***	***|saul'yev-Gp26|***	***|ditermenistic-Gp26|***
0	0	0	0
0.1	0.1	0.01575	0.01653
0.2	0.2	0.04425	0.04100
0.3	0.3	0.06718	0.04256
0.4	0.4	0.09982	0.09902
0.5	0.5	0.12078	0.12041
0.6	0.6	0.14600	0.12727
0.7	0.7	0.15877	0.11709

0.8	0.8	0.08448	0.16523
0.9	0.9	0.08089	0.08428
1	1	0	0

Example 3 Let we try to find the approximation solution of the SPDE with multiplicative noise.

$$dU_t(x) = \kappa\Delta U_t(x)dt - U_t(x)dW_t(x) \qquad (65)$$

With

$$U_0(x) = x^2(1-x^2) \ \ and \ \ U_t(0) = U_t(1) = 0 \qquad (66)$$

$for\ x \in (0,1), t \in [0,T)$,where $W_t(x)$ is space-time white noise wiener process, and where κ is a small parameter, we will have $\kappa = \frac{1}{1000}$.

By using Doss-Susmann transform (30). We find

$$dv(t,x) = 0.001\frac{\partial^2 v}{\partial x^2}(t,x)dt + \left(0 - \frac{1}{2}\sum_{l=1}^{n}\left(h^l(t)\right)^2\right)v(t,x)dt \qquad (67)$$

With

$$\mathcal{A}U = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\frac{\partial^2 U}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i\frac{\partial U}{\partial x_i} = \sum_{i,j=1}^{1}\frac{1}{1000}\frac{\partial^2 U}{\partial x_i \partial x_j} + \sum_{i=1}^{1} 0\frac{\partial U}{\partial x_i} = \frac{1}{1000}\frac{\partial^2 U}{\partial x^2} \qquad (68)$$

Then $b(X_t) = 0\ and\ \sigma_{ij} = \sqrt{0.002}$,and the Markov process X was governed by the infinitesimal generator A of this stochastic differential equation is:

$$X_t = x_0 + \int_0^t \sqrt{0.002}\,dB_s \ \ ,0 \le t \le T \qquad (69)$$

where the initial condition x_0 in $\mathbb{R}^{\boldsymbol{d}}$. if $h(t) = 1$ and $n = 1$ then (67) becomes :

$$\frac{dv(t,x)}{dt} = 0.001\frac{\partial^2 v}{\partial x^2}(t,x) + \frac{1}{2}v(t,x) \qquad (70)$$

With

$$v_0(x) = x^2(1-x^2) \ \ and \ \ v_t(0) = v_t(1) = 0 \qquad (71)$$

$for\ x \in (0,1), t \in [0,T)$

Now find the numerical solution of the partial differential equation (PDE)(65) by using accelerated genetic algorithm. We found at generation 10 that :

$$Gp10(t,x) = 2exp(-2exp(t))\,sinx^2 \qquad (72)$$

And the solution of the stochastic ordinary differential equation (69) (Markov process) generated by the infinitesimal generator $\mathcal{A}$ by accelerated genetic algorithm is :

$$X_t = x_0 + \sqrt{2}B(t) \qquad (73)$$

Then , the solution of the original equation (65) is obtained by substituting (72),(73) in equation (29):

$$U(t,x) = e^{W_t(x)} \times Gp10(t,X_t) = e^{W_t(x)} \times 2\exp(-2\exp(t))\sin\left(x_0 + \sqrt{0.002}B(t)\right)^2 \qquad (74)$$

Fig.5 show this solution

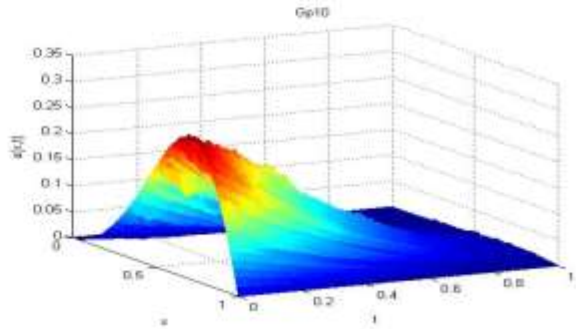

Fig.5 solution of SPDE (65)

And then compared this solution by our method with the solution obtained by Saul'yev method and with its corresponding deterministic solution. This comparison shown in
Fig. 6 below :

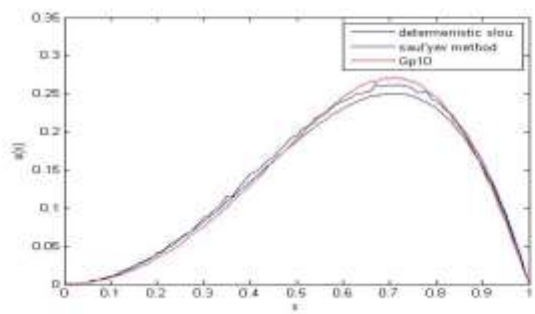

Fig.6 comparison of solutions of SPDE (65)

The comparisons of errors of these solutions was shown in table (4.2).

Table (4.2) Comparisons of the errors.

t	*x*	*|saul'yev-Gp10|*	*|ditermenistic-Gp10|*
0	0	0	0
0.1	0.1	0.00061	0.00131
0.2	0.2	0.00249	0.01378
0.3	0.3	0.04720	0.04396
0.4	0.4	0.09117	0.08608
0.5	0.5	0.14238	0.13393
0.6	0.6	0.03907	0.17049
0.7	0.7	0.22878	0.20517
0.8	0.8	0.22330	0.19364
0.9	0.9	0.15338	0.12285
1	1	0	0

CONCLUSIONS

Application of a new technique for solving stochastic partial differential equations. Such as applied of accelerated genetic algorithm (AGA) to find the numerical solutions of stochastic partial differential equations with additive and multiplicative cylindrical Brownian motion (or space-time white noise) , using Doss-Susmann transformation , to transform these equation into partial differential equations and stochastic ordinary differential equation , then applied the AGA to find the numerical solutions of transformed equations and then the solution of original equations. We noted that this method has general utility for applications , and we found that insertion of boundary condition as a chromosomes in the population quick the algorithm to approximate the numerical solutions.

In order to compare the results that have been obtained by using accelerated genetic algorithm , validating, it has comparison with some numerical methods (such as finite difference method and the saul'yev method), where these methods are used to solve this kind of stochastic partial differential equations and it's always convergence. It turns out that the results that have been obtained by using accelerated genetic algorithm are good results and convergence with these methods.

The main problem that we faced during the application of the (AGA) to find numerical solutions of stochastic differential equations , are noise-generating process, such as (Brownian motion or cylindrical Brownian motion). Where the values of the noise must be normally distributed with zero mean and variance equal to dt i.e. $N(0, dt)$. To achieve this value of dt must be very small change so that we get the largest number of values within the specified interval , these issues that affect on the shape and distribution of the noise and shows its influence is clear in the final solutions.

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